Heat Conduction with Solidification in a Stratified Medium

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With the use of the heat balance integral method, a formula for the depth of solidification in a stratified medium is derived that applies to Newton's cooling at the surface, where the temperature of the cooling medium is variable. The presence of a surface layer with zero latent heat is given a special consideration. Comparison of the proposed formula with the direct field measurements and the calculations carried out by others shows a satisfactory coincidence. For the simplified boundary conditions, the formula reduces to a known analytical solution.

Over the years both applied mathematicians and engineers strived to establish methods for predicting the depth of solidification (Stefan's problem). Consequently, several lines of thought can be distinguished here. In the more rigorous approach, the general solution of the governing differential equation with the corresponding boundary conditions is sought. For the present problem such a solution can be stated only in terms of nonlinear integral equations (*1, 2*), whose evaluation still requires much mathematical effort. Important, however, remains the exact solution of a simplified version of the problem by Neumann (*3*).

Simplified analytical methods may be considered legitimate if more rigorous analysis fails to produce satisfactory results (*4* to *6*). Using this kind of approach, I obtained an approximate solution for the freezing of a semi-infinite slab with Newton's cooling at the surface (*7*), which was bounded analytically. In this paper, an attempt is made to derive formulas applicable to a generalized stratified medium. The results are compared with the field measurements (*8*) and with the solutions of the corresponding differential equations obtained with the help of analog and digital computers (*9, 10*). The necessary thermal properties are obtained from reference *11*. Furthermore, the calculation of the initial frost penetration is compared with the analytical results outlined in reference *12*.

STATEMENT OF THE PROBLEM

The medium with the variable thermal properties undergoing the process of solidification is approximated here by a semi-infinite slab consisting of m layers, the m^{th} layer extending to infinity. The thermal properties and the moisture content therefore change discontinuously from one layer to another, but remain constant within each layer. There is one exception: When the freezing temperature T_f is reached, the thermal conductivity suddenly changes. At the time 0_+, the slab, originally at a uniform temperature T_∞, is suddenly exposed to a cooling medium at the temperature T_a. Consequently, Newton's cooling results. In addition, it is assumed that the expansion of the solidified layer is negligible here (because the voids of the medium are only partly filled with moisture), and that the effects of moisture migration may be neglected.

From Figure 1 it may be seen that for the j^{th} layer at the total depth D_{j-1}, the corresponding boundary value problem becomes

$$\frac{\partial T}{\partial t} = a_{i,j}\frac{\partial^2 T}{\partial x^2}, \qquad D_{j-1} \leqq x \leqq D_j \tag{1}$$

with $i = 1, 2$ for each j, where $1 \leqq j \leqq m$, and where

$$T(x, t) = T_\infty, \qquad x \geqq 0,\ t < 0 \tag{2a}$$

$$-k_{i,1}\frac{\partial T_{i,1}}{\partial x} = h(T_a - T_s),\ t > 0_+, x = 0 \tag{2b}$$

$$T_{1,j}(\xi_j, t) = T_{2,j}(\xi_j, t) = T_f \tag{2c}$$

$$-k_{1,j}\frac{\partial T_{1,j}}{\partial x} = -k_{2,j}\frac{\partial T_{2,j}}{\partial x} - L_j\frac{d\xi_j}{dt}, x = \xi_j \tag{2d}$$

Also, both temperature and heat flow must be continuous at the boundaries of each layer.

Because of the nonlinear boundary condition, Equation (*2d*), the whole problem becomes nonlinear; the heat balance integral method will now be applied to find a solution. This method requires that the temperature profile be assumed as a function of a finite thermal penetration depth x_t; here, in particular, a distinction between the temperatures of the frozen and the liquid zones has to be made also. Then if in the process of solidification the sensible heat of the frozen zone is a relatively small portion of the total heat to be removed, the temperature of that zone may be represented by a continuous broken line:

$$(T_{1,j} - T_a)/\theta_1 = \frac{(B_{j-1} + x)k_{1,j}}{(B_{j-1} + \xi_j)k_{1,j}}, \quad D_{j-1} < x \leqq \xi_j$$

$$(T_{1,j} - T_a)/\theta_1 = \frac{(B_{j-2} + x)k_{1,j}}{(B_{j-1} + \xi_j)k_{1,j-1}}, D_{j-2} < x \leqq D_{j-1} \tag{3}$$

etc., and $T_{1,j} = T_f$, $x \geqq \xi_j$. Here, $B_{j-1} = k_{1,j}\,R_{j-1} - D_{j-1}$, where R_{j-1} is the combined steady state thermal resistance of the first $j - 1$ layers and of the heat transfer coefficient.

In the liquid zone, temperature is approximated by a parabola

$$(T_{2,j} - T_f)/\theta_2 = 2\frac{x - \xi_j}{x_t - \xi_j} - \frac{(x - \xi_j)^2}{(x_t - \xi_j)^2} \tag{4}$$

for $\xi_j < x \leqq x_t$, and $T_{2,j} = T_\infty$, $x \geqq x_t$. A parabolic temperature profile is sufficient to represent here only a rough outline of the actual temperature distribution; this approach appears, however, satisfactory in calculating the

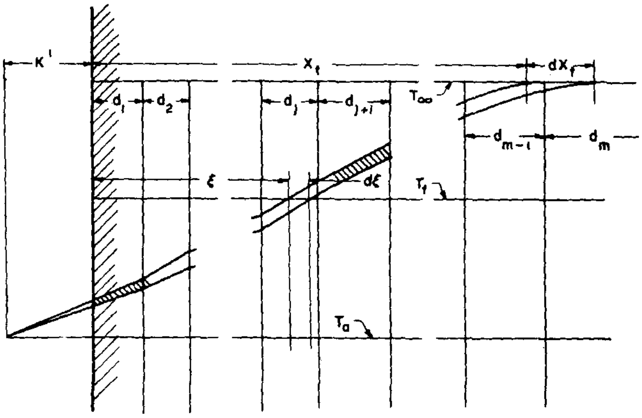

Fig. 1. Schematic representation of solidification in a stratified medium.

sensible heat of that zone according to the heat balance integral method. It may be noted that Equations (3) and (4) satisfy all boundary conditions, except that when $x = x_t$, $T_2 = T_\infty$, which represents some distortion of the true temperature profile. The expression for the heat involved in the freezing of the j^{th} layer ($j \leq m$) is according to Figure 1

$$Q_j = \int_0^\infty C_1(x) T_{1,j}\, dx + \int_{\xi_j}^\infty C_2(x)(T_{2,j} - T_f)dx + L_j(\xi_j - D_{j-1}) + \sum_{j+1}^{m} L_n d_n \quad (5)$$

This expression for Q_j will be approximately correct if the actual volumetric specific heats are replaced by the mean values

$$\overline{C}_{1,j} = \frac{\sum_{n=1}^{j-1} d_n C_{1,n} + \frac{1}{2} d_j C_{1,j}}{\sum_{n=1}^{j-1} d_n + \frac{1}{2} d_j} \quad (6)$$

and

$$\overline{C}_{2,j} = \frac{\sum_{n=j+1}^{k} d_n C_{2,n} + \frac{1}{2} d_j C_{2,j}}{\sum_{n=j+1}^{k} d_n + \frac{1}{2} d_j} \quad (7)$$

The index k in Equation (7) will depend on the index of the layer to which the effects of surface cooling have penetrated.

All the energy set free by the freezing process must leave the slab according to the Fourier law of heat conduction:

$$\frac{d}{dt}(Q_j) = -k_{1,1} \frac{\partial T_{1,j}}{\partial x}\bigg|_{x=0} \quad (8)$$

With the substitutions $\alpha_j = \theta_2 \overline{C}_{2,j}/\theta_1 \overline{C}_{1,j}$, $\mu_j = \theta_1 \overline{C}_{1,j}/L_j$, and ϵ_j, this gives the result

$$\left[\frac{1}{2}(1 - \epsilon_j)\mu_j + \alpha_j \mu_j \left(\frac{1}{3}\frac{dx_t}{d\xi_j} + \frac{2}{3} \right) + 1 \right] L_j \frac{d\xi_j}{dt} = \theta_1 k_{1,j} (B_{j-1} + \xi_j)^{-1} \quad (9)$$

The new term ϵ_j is of the order $(B_{j-1} + \xi_j)^{-2}$, and $|\epsilon_j| < 1$. Also, $\epsilon_1 \to 0$ as $h \to \infty$. In the present case, where the presence of insulation on top of the slab makes the beginning of the freezing process independent from the latent heat effects, the approximation $\epsilon_j \approx 0$ can be made for $j \neq 1$.

Now let the expression in the square brackets in Equation (9) be identically equal to the term ν_j^{-2} assumed *for convenience to be independent of* ξ_j *within* a given layer; ν_j is a dimensionless parameter that is a function of $dx_t/d\xi_j = x'$. Then, the above condition on ν_j implies that x' is also independent of ξ_j. Conversely, from combining Equations (9) and (2d), an expression for x' is found in terms of ν_j and $\delta_j^2 = k_{1,j}\overline{C}_{2,j}/k_{2,j}\overline{C}_{1,j}$. After elimination of x' one obtains the relation

$$\nu_j^4 \left(1 + \frac{1}{2}\mu_j + \alpha_j \mu_j \right) - \nu_j^2 \left(2 + \frac{1}{2}\mu_j + \alpha_j\mu_j + \frac{2}{3}\frac{\alpha_j^2 \mu_j}{\delta_j^2} \right) + 1 = 0 \quad (10)$$

from which ν_j can be determined. Equation (9), after integration and a few simple transformations, yields the formula

$$\xi_j' = -k_{1,j} R_{j-1} + \left[k^2_{1,j} R^2_{j-1} + \nu_j^2 \frac{2k_{1,j}\theta_1}{L_j}(t - t_{j-1}) \right]^{1/2} \quad (11)$$

where $\xi_j' = \xi_j - D_{j-1}$ is the local coordinate for the freezing front in the j^{th} layer. The time $t_j - t_{j-1}$ is the time span necessary for θ_1 to achieve the result $\xi_j' = d_j$; when $t = t_{j-1}$, the solidification in the j^{th} layer is just about to start. Equation (11) also may be conveniently expressed in terms of the freezing index $I_j = \theta_1 (t - t_{j-1})$.

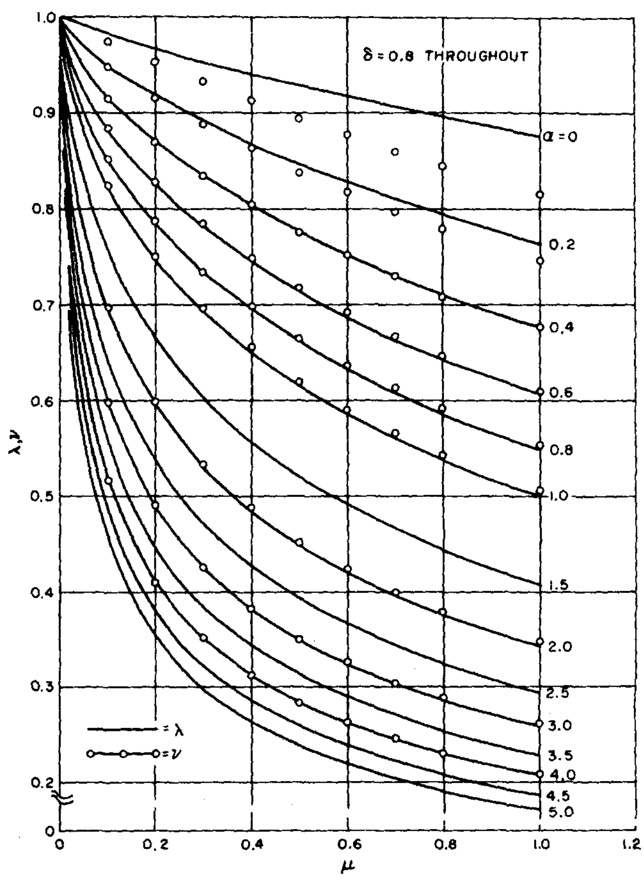

Fig. 2. Comparison of λ and ν.

The condition $0 < \nu_j < 1$ uniquely determines the root of Equation (10) that must be used together with Equation (11) to determine the depth of solidification. Equation (10) is identical with the equation defining ν in reference 7, except that now the parameters α_j, μ_j, and δ_j are defined in terms of the mean volumetric specific heats $\overline{C}_{1,j}$ and $\overline{C}_{2,j}$. For $j = 1$, Equation (11) reduces to the one-layer solution of reference 7, and for $h \rightarrow \infty$ it can be directly compared with the Neumann's exact solution for a semi-infinite slab originally at T_∞, whose surface is suddenly exposed to a freezing temperature θ_1:

$$\xi = \lambda \left[\frac{2\theta_1 k_1 t}{L} \right]^{1/2} \tag{12}$$

Thus, for the boundary conditions specified above, Equation (11) reduces to a well-known exact solution, the only difference being that λ is used instead of ν. Whereas λ is obtained from a transcendental equation, ν is the root of an algebraic equation that is much more convenient to solve. From the direct comparison λ and ν in Figure 2, it is seen that they do not deviate much numerically from each other, being the functions of the same three parameters α, δ, and μ.

EFFECT OF VARIABLE AIR TEMPERATURE

A physically justifiable expression for the effect of the variable air temperature appears to be the mean integral value of that temperature. Since the air freezing index I, obtainable from the meteorological reports, approximates the integral of freezing temperature taken over the freezing season t_s, the result of the operation

$$\overline{\theta}_1 = \frac{1}{t_s} \int_0^{t_s} (T_f - T_a)\,dt \tag{13}$$

will provide a convenient reference freezing temperature $\overline{\theta}_1$. Because the influence of short-term air temperature variations is damped out quite rapidly at depths for which the times of penetration of surface effects are of a higher order of magnitude than the period of the surface temperature fluctuations, it is fair to assume that $\overline{\theta}_1$ will adequately represent the variable air temperature at the freezing front $x = \xi$. In particular, from Figure 2 it may be observed that ν is a relatively weak function of the excess surface temperature: a given change in θ_1 will result in changes of α and μ whose cumulative effect on ν will be slight. Consequently, for the case of a variable air temperature, the pertinent parameters appearing in Equation (11) may with some justification be based on $\overline{\theta}_1$.

INITIAL SURFACE LAYER FREEZING

An expression still must be derived for the case where the freezing temperature first has to penetrate through the layer $d_1 = l$ that is free from latent heat effects (equivalent to a layer of insulation or pavement on the top of wet soil). For the sinusoidal variation of surface temperature, initial freezing with $L = 0$ is very simple analytically. For such a temperature variation, a two-layer solution has been obtained by Lachenbruch (*13*). Since the onset of the freezing season is connected ordinarily with a sudden temperature change, a step temperature change will be the basis of the following formula for the initial freezing of the surface layer. In the derivation, the heat balance integral method will be used. With a few rather obvious changes, the present approach could be also applied to the freezing of a buried layer of insulation in a wet soil. For the boundary conditions, Equations (2*a*) and (2*b*) apply.

The problem is outlined schematically in Figure 3. At $x = l$ a new temperature T_p is introduced. The first stage of freezing takes place when the step temperature change is not yet felt at the bottom of the surface layer, and the temperature is represented by a straight line

$$T_{1,0} = \frac{x + k'}{x_p + k'} (T_\infty - T_a) + T_a, x < x_p \tag{14}$$

Here, x_p is the thermal penetration depth. The relation analogous to Equation (8) yields, with $a_p = k_1/C_1$

$$t_{p,1} = \frac{1}{4a_p} \left[l^2 + 2lk' - 2(k')^2 \log \left(\frac{k' + l}{k'} \right) \right] \tag{15}$$

When the cooling effect has reached the bottom of the surface layer, it is convenient to define again two temperatures:

$$T_{1,0} = \frac{x + k'}{l + k'} (T_p - T_a) + T_a, x \leqq l \tag{16}$$

$$T_{2,0} = 2(T - T_p) \left[\frac{x - l}{x_t - l} - \frac{1}{2} \frac{(x - l)^2}{(x_t - l)^2} \right] + T_p, x \leqq x_t \tag{17}$$

with $T_{1,0} = T_p$, $x \geqq l$, and $T_{2,0} = T_\infty$, $x \geqq x_t$. Then, applying Equation (8) to the present situation and using the condition of continuity for heat flow at $x = l$, one gets the formula

$$t_{p,2} = \frac{(l + k')^2}{4a_p} \left\{ 2 \left[1 - \frac{(k')^2}{(l + k')^2} \right] \log \left(1 + \frac{\theta_2}{\theta_1} \right) - \frac{8}{3} \frac{\overline{C}_{2,2} k_{2,2}}{C_p k_p} \log \left(1 + \frac{\theta_2}{\theta_1} \right) + \frac{4}{3} \frac{\overline{C}_{2,2} k_{2,2}}{C_p k_p} \left[\left(1 + \frac{\theta_2}{\theta_1} \right)^2 - 1 \right] \right\} \tag{18}$$

with the total time $t_p = t_{p,1} + t_{p,2}$ required for the freezing temperature to penetrate through the surface layer.

DISCUSSION OF THE RESULTS

After the formulas for freezing of a stratified medium have been derived, it is interesting to discuss their implications, drawing also on the results of other investigators. It has been shown already that for one layer only and $h = \infty$, Equation (11) reduces to Neumann's solution. By rearranging Equation (11), the partial freezing index in degree days necessary for solidification in the j^{th} layer is obtained as

$$I_j = \frac{L_j \xi_j'}{24\nu_j^2} \left(R_{j-1} + \frac{\xi_j'}{2k_{1,j}} \right) \tag{19}$$

Equation (19) is already in the form which makes it most convenient for calculations if the total freezing index is known. With $\nu = 1$, and when no distinction is made between the properties of the frozen and the liquid zones, it reduces to the so-called *Stefan's equation* adapted for the freezing of a stratified medium (see, for example, reference 8).

A semiempirical formula for stratified medium proposed by Aldrich (*9*) has the same form as Neumann's solution, but it uses an effective value of L/k, based on the estimated depth of freezing. In Aldrich's formula, no distinction is made between the air and the surface temperatures, and the mean of the frozen and the liquid values is used for thermal properties. The parameters α and μ are determined from the averaged values of L and C of the layers

TABLE 1. PARAMETERS USED IN EXAMPLE 1 (SITE A)

Layer No.	d	ρ	u	K	$\overline{C}$	L
1	0.17	130	—	0.83	22	—
2	0.33	135	0.04	1.33	24	775
3	1.00	125	0.057	1.12	25	1,020
4	1.00	115	0.104	0.92	25	1,720
5	1.00	115	0.098	0.75	25	1,620
6	1.00	115	0.093	0.85	25	1,540
7	0.49	110	0.122	0.88	25	1,920

Total: 4.99 ft.

involved. Quinn and Lobacz (*14*) combined Aldrich's method with the Lachenbruch's formula for the initial effects of freezing. The main drawback of this formula is that it is explicitly based on the (generally unknown) surface temperature.

To account for the effects of the sensible heat and for the difference between the temperature of the air and that of the surface, the modified Stefan's equation has been used sometimes with a reduced freezing index to achieve a better correlation with observed results. Thus Braun (*8*) has used a correction factor of 0.74, obtained empirically. Braun's field measurements, plus the results obtained by Aldrich on an analog computer (*9*), and by Seider and Churchill (*10*) and Churchill (*12*) on a digital computer, will be used for comparison to justify the physical reasoning used in this paper and all the simplifications that have been necessary.

NUMERICAL EXAMPLES

The first example refers to the case of frost penetration through bituminous pavement on U.S. Highway 63, 9 miles from Rochester, Minnesota, during the freezing season 1956-57. The thermal conductivities are determined according to the information on density, moisture content, and texture of the soil with the help of reference *11*, which also suggests for the specific heats the relations $C_1 = \rho\ (0.17 + 0.5u)$ and $C_2 = \rho\ (0.17 + u)$. The latent heat of fusion is $L = 143.4\ u$, which disregards the effect of water bound by the finely dispersed soil particles.

The data are shown in Table 1. The total freezing index $I = 1{,}516$ degree days has been accumulated in 114 days; $T_\infty = 45°$F. is obtained from U.S. Weather Bureau reports on Rochester, Minnesota. For the heat transfer coefficient, the value $h = 6.0$ B.t.u./(hr.)(sq.ft.)(°F.) is representative for heat transfer to building materials for wind velocity less than 15 miles/hr. The approximation $\alpha \approx \theta_2/\theta_1$ and $\delta \approx 1$ is applied in this and in the next example. The result of the calculation by Equation (11) is $\xi = 4.99$ ft. Braun's measurement gives $\xi = 5.2 \pm 0.1$ ft. and his calculation $\xi = 5.65$ ft., whereas by Aldrich's method $\xi = 5.01$ ft.

The additional calculations by Equation (11) are compared with Braun's measurements and calculations in Table 2. It is seen that the results by the method proposed in this paper are in a reasonable agreement with field measurements.

Next, the present method is compared with the hydraulic analog computer solutions for certain assumed values of soil properties. For this example, $I = 1{,}568$ degree days accumulated in 158 days, and $T_\infty = 37°$F.; the remaining parameters are shown in Table 3. Aldrich calculates here $\xi = 5.6$ ft. by his method (*9*) and the analog computer gives $\xi = 5.7$ ft. for a step surface temperature change and $\xi = 5.8$ ft. for a sinusoidal change of surface temperature. Equation (11) gives here $\xi = 5.6$ ft., which agrees within 3% with the corresponding computer solution.

TABLE 2. COMPARISON OF CALCULATIONS BY EQUATION (11) WITH ADDITIONAL FIELD MEASUREMENTS

Site	Calculation [Equation (11)], ft.	Calculation (Braun), ft.	Measurement (Braun,) ft
C	5.00	5.15	5.1
D	4.44	4.65	4.5
E	4.78	4.68	4.6
F	4.81	4.82	5.1
K	5.15	5.35	5.2

More recently, Seider and Churchill published results of digital computer calculations on the effect of surface insulation on freezing in a homogeneous medium (*10*). The depth of frost penetration is given as a function of $N_{Fo} = \pi a_p t/l^2$, for $\theta_2/\theta_1 = 0.2$, $L/C_{1,2}\theta_1 = 2.34$, $a_{1,2}/a_{2,2} = 1.5$, $a_{1,2}/a_p = 14.5$, $k_{1,2}/k_p = 12.0$, and $k_{1,2}/k_{2,2} = 1.0$.

From Figure 5 of reference 10 it is seen that for $N_{Fo} = 2.2$, freezing temperature reached the bottom of the insulation and from Figure 3 for $N_{Fo} = 25.15$ frost penetration in the earth is 2.36 times the thickness of the insulation l. Here, Equations (15) and (18) give $N_{Fo} = 2.25$ for the frost penetration through l; there remains still $N_{Fo} = 22.9$ for freezing of the underlying layer. By the method of this paper, the remaining calculations proceed as follows. First, with $z = \xi_2'/l$ Equation (19) is transformed to read

$$\pi(t_2 - t_1)\ a_p/l^2 = \frac{\pi L_2 a_p k_{1,2} z}{\theta_1 C_{1,2} a_{1,2} k_p \nu_2^2} \left[1 + \frac{z}{2} \frac{k_p}{k_{1,2}} \right]$$

Then, ν_2 is evaluated from Equation (10), and from the substitution of all the available data, $z = 2.39$, which compares well with the computer solution. The broken line temperature profile [Eq. (3)] also compares well with the temperature distribution shown in Figure 3 of reference *10*.

In Figure 4 the effect of insulation on the freezing front location in the form of the relative freezing depth ξ^* as a function of N_{Fo} is shown. The calculated values of ξ^* are 0.014 and 0.15, which should be compared with the values 0.02 and 0.14 scaled off the figure, for $N_{Fo} = 3.0$ and 10, respectively. The effect of insulation on the dimensionless heat flux density Q^* to the surface is shown in Figure 5. Of special interest is the behavior of the approximate solution immediately after the freezing process started. Thus, for $N_{Fo} = 2.2$, Q^* calculated is 0.269 vs. $Q^* = 0.27$ scaled off the figure. At N_{Fo} 2.65, these values are, respectively, 0.318 and 0.315.

The approximate calculation of the time required for the initial frost penetration will be compared with the

TABLE 3. PARAMETERS USED IN EXAMPLE 2

Layer No.	d, ft.	Description	K	C	L
1	0.25	Bit. concrete	0.8	28	—
2	0.50	Base course	1.0	23	850
3	1.79	Subbase	1.30	25	1,200
4	3.06	Subgrade	1.70	27	2,900

Total: 5.6 ft.

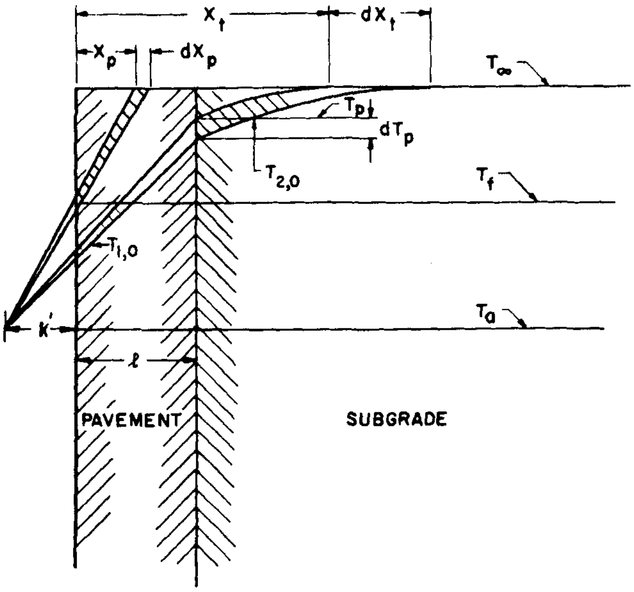

Fig. 3. Penetration of freezing temperature through a dry surface layer.

exact analytical solution prepared by Churchill (*12*). For the specific situation described in Figure 4 of reference *12*, for $\theta_2/\theta_1 = 0.25$, $N_{Fo} = 100$ is required for the initial frost penetration to take place, whereas from Equations (15) and (18) it is $N_{Fo} = 92.3$. This appears to be a close enough agreement for making estimates of the initial delay of freezing due to the presence of a dry layer at the surface.

Finally, the following generalization can be made. Since the cases considered in this discussion were taken at random from the existing literature, the results obtained thus far get more significance, since they are based on a process resembling random sampling. It is conceivable however, that under special circumstances the approximate analytical method, proposed here, will show a poorer accuracy than the results above would indicate. Consequently, where legitimate doubts arise about the advisability of applying approximate methods, analytical bounds and/or numerical methods must be considered.

CONCLUSIONS

In this study formulas have been derived for calculation of the depth of freezing in a multilayer medium, with a method for estimating the time necessary for the initial penetration of the freezing temperature through a dry surface layer. The proposed formulas are applicable for Newton's cooling at the surface and can be used in connection with the air freezing index. The calculated examples have shown a satisfactory agreement with the field data and with the solutions furnished by the analog and digital computers.

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NOTATION

a = thermal diffusivity, sq.ft./hr.
C = volumetric specific heat, B.t.u./(cu.ft.) (°F.)
$\bar{C}$ = mean value of C
d = thickness of layer, ft.
D_j = partial sum of j layers, ft.
h = heat transfer coefficient, B.t.u./(hr.) (sq.ft.) (°F.)
I = freezing index, °F. day
k = thermal conductivity, B.t.u./(hr.) (ft.) (°F.)
k' = k/h, ft.
l = thickness of dry surface layer, ft.
L = volumetric latent heat of fusion, B.t.u./cu.ft.
R = thermal resistance, (hr.) (sq.ft.) (°F.)/B.t.u.
t = time, hr.
T = temperature, °F.
u = moisture content, $lb._{H_2O}/lb.$

Greek Letters

α = $\theta_2\bar{C}_2/\theta_1\bar{C}_1$, dimensionless sensible heat
δ = $k_1\bar{C}_2/k_2\bar{C}_1$, dimensionless thermal diffusivity
θ_1 = $T_f - T_a$, reduced temperature, frozen zone, °F.
$\bar{\theta}_1$ = mean integral value of θ_1, °F.
θ_2 = $T_\infty - T_f$, reduced temperature, liquid zone, °F.
$\bar{\theta}_2$ = mean integral value of θ_2, °F.
λ = sensible heat coefficient, Neumann's solution
μ = $\theta_1\bar{C}_1/L$, reciprocal dimensionless latent heat
ν = sensible heat coefficient, root of Equation (10)
ξ = depth of solidification, ft.
ξ_j' = $\xi_j - D_{j-1}$, local coordinate of solidification in the j^{th} layer, ft.
ρ = dry density of the medium, lb./cu.ft.
ϵ = term in Equation (9)

Subscripts

1 = frozen zone
2 = liquid zone
∞ = condition far away from surface
a = condition of the cooling medium (air)
f = condition where phase changes
p = pavement or insulating layer
n, m = first index refers to zone, second index refers to layer, except for freezing of the dry surface layer, where 1, 0 applies for $x < l$ and 2, 0 applies for $x > l$

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